# CONVECTIVE INSTABILITY SPECTRUM IN A VERTICAL CHANNEL WITH PERMEABLE BOUNDARIES

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In an earlier paper [1] we have solved the problem of emergence of stationary plane-parallel convection in a fluid filling a plane vertical channel with permeable boundaries and heated from below. We have shown that the characteristic values of the Rayleigh number which determine the stability limits with respect to stationary perturbations, depend on the velocity of transverse motion of the fluid, and with increasing the Péclet number the "closure" of the adjacent levels of the instability spectrum ensues. Thus, the stationary convection can take place only if the transverse velocity does not exceed a certain limit. Discussing this conclusion, we have suggested [1] that the confluence of stationary motion levels is accompanied by the emergence of oscillatory convective motions. In the present paper we shall produce the results of a numerical analysis of nonstationary perturbation spectra. It follows from these results that with increasing the Péclet number, convective motion of stationary oscillation nature occurs. Hence, depending on the parameter (Peclet number), the basic state (transverse motion of the fluid) becomes unstable with respect to either monotonic or oscillatory perturbations. The analysis of spectra makes it possible to determine the limits of regions of the monotonic and oscillatory instabilities.

**1.** Let us assume a transverse motion with a constant uniform velocity $v_0$ in a plane vertical channel with permeable boundaries $x = \pm h$ ; heating from underneath causes a linear height distribution of temperature with a gradient $A$. Let us consider perturbed motion described by

$$v_x = v_0, \quad v_y = 0, \quad v_z = v(x, t); \quad T = -Az + \theta(x, t); \quad p = p(z, t) \tag{1.1}$$

Assuming that the perturbations are exponentially time dependent in accordance with $\exp(-\lambda t)$, the general equations of convection yield the following amplitude equations:

$$-\lambda v + \frac{a}{P} v' - v'' - R\theta - C = 0 \tag{1.2}$$

$$-\lambda P\theta - v + a\theta' - a\theta'' = 0 \tag{1.3}$$

In the above $v(x)$ and $\theta(x)$ are amplitude parts of the perturbations and $C$ is the variable separation constant. The equations are set in dimensionless variables: the units of distance, time, velocity and temperature are taken as $h$, $h^2/\nu$, $\chi/h$ and $Ah$, respectively. The numbers of Rayleigh $R$, Péclet $a$ and Prandtl $P$ are determined as follows:

$$R = \frac{g\beta A h^4}{\nu\chi}, \quad a = \frac{v_0 h}{\chi}, \quad P = \frac{\nu}{\chi}$$

The amplitudes satisfy the uniformity conditions

$$v(\pm 1) = 0, \quad \theta(\pm 1) = 0, \quad \int_{-1}^{1} v\,dx = 0 \tag{1.4}$$

The boundary value problem (1.2)–(1.4) determines characteristic perturbations and their decrements $\lambda$. Real $\lambda$ correspond to monotonically damping or increasing with time perturbations. The stability limit with respect to these perturbations is found from the condition $\lambda = 0$. Complex decrements $\lambda = \lambda_r + i\lambda_i$ correspond to oscillatory perturbations; the stability limit is defined from the condition $\lambda_r = 0$, and $\lambda_i$ yielding the dimensionless frequency of oscillations (subscripts denote real and imaginary parts).

**2.** We shall solve our problem by means of Bubnov-Galerkin method. Let us write the amplitude functions as series expansions

$$v = a_0 v_0 + a_1 v_1 + \ldots + a_N v_N, \qquad \theta = b_0\theta_0 + b_1\theta_1 + \ldots + b_M\theta_M \qquad (2.1)$$

where $v_n$ and $\theta_m$ are basic functions satisfying the conditions (1.4). It will be convenient to take for the basis the eigenfunctions of the problem of stationary plane-parallel convection in a vertical channel with impermeable boundaries [2]

$$v_n = \frac{\cos r_n x}{\cos r_n} - \frac{\operatorname{ch} r_n x}{\operatorname{ch} r_n} \quad (n = 0, 2, 4\ldots), \quad v_n = \sin\frac{(n+1)\pi}{2} \quad (n = 1, 3, 5\ldots) \qquad (2.2)$$

$$\theta_m = \frac{\cos r_m x}{\cos r_m} + \frac{\operatorname{ch} r_m x}{\operatorname{ch} r_m} - 2 \quad (m = 0, 2, 4\ldots), \quad \theta_m = \sin\frac{(m+1)\pi}{2} \quad (m = 1, 3, 5\ldots)$$

where $r_n$ are the roots of the transcendental equation

$$\operatorname{tg} r = \operatorname{th} r, \qquad r_0 = 3.927, \qquad r_2 = 7.069, \ldots$$

Substituting series (2.1) into the left parts of (1.2) and (1.3), multiplying by $v_i$ and $\theta_k$, respectively, and integrating in $x$ from −1 to 1, we obtain a set of homogeneous linear equations for the expansion coefficients $a_n$ and $b_m$. The condition that must be satisfied if the above set of equations is to have a nontrivial solution, namely that its determinant equals 0, defines the spectrum of eigenvalues $\lambda$. Computations were carried out for $N = M = 7$. The eigenvalues of the 16th order matrix were calculated with a computer using the orthogonal-step method.

**3.** Let us now discuss our results. Characteristic decrements $\lambda$ depend on three parameters, viz. the Rayleigh number $R$. The Peclet number $a$ and the Prandtl number $P$. Our main interest was in investigating qualitative features of the decrement spectrum structure, in connection with the earlier discovered closure of instability levels. All our computations were, therefore, carried out for a fixed value of the Prandtl number, $P = 1$.

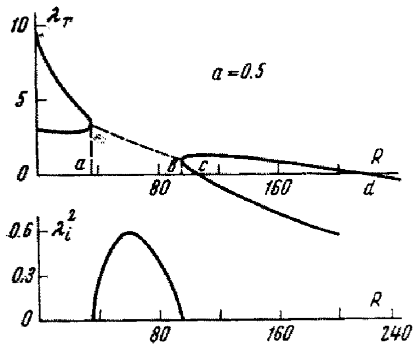


Fig. 1

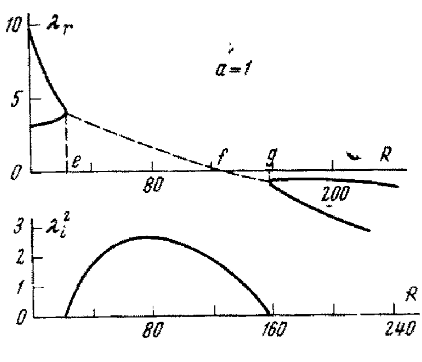


Fig. 2

Figures 1 and 2 represent the dependence of the real and imaginary parts of decrements of the two "lower" perturbation modes on the Rayleigh number for $a = 0.5$ and $a = 1$. According to earlier results [1], the first of these values of the Péclet number is in the region in which stationary motions take place; there are no stationary motions for $a = 1$ . Comparison of the spectra in Fig. 1 and 2 explains the nature of the process of "closure" of neutral lines of stationary perturbations.

It can be seen in Fig. 1 that for small values of the Rayleigh number both decrements are real and positive, and corresponding perturbations damp monotonically. At the point $a$ a confluence of the two real spectrum levels occurs and a pair of complex conjugate decrements corresponding to oscillatory perturbations is generated. In the interval $(a, b)$ the damped oscillations exist (common real part $\lambda_r$ is shown dashed, $\lambda_r > 0$ ). At the point $b$ the pair of complex conjugate decrements splits again into two real decrements. To the right of $b$ only monotonic perturbations are possible; their real decrements change signs consecutively in points $c$ and $d$, generating two levels of monotonic instability . The corresponding Rayleigh numbers determine the neutral points of monotonic perturbations or, in other words, the points of existence of stationary convection (the critical Rayleigh numbers have practically precise values [1]). Thus, for $a = 0.5$, the only instability possible is that with respect to monotonic perturbations.

For $a = 1$ (Fig. 2), as the number $R$ increases, there is also at first confluence of real levels into a complex conjugate pair (oscillatory perturbations onset point $e$) and then this pair splits into two real decrements (point $g$). In this case, however, oscillatory perturbations are damped only in the interval $(e, f)$. At the point $f$ the real part of decrements vanishes and this is the neutral point of oscillatory perturbations. At the proper value of the Rayleigh number, convection in the form of stationary oscillations is superposed on the fundamental transverse motion (the frequency of neutral oscillations can be found from the graph of the imaginary part of decrement $\lambda_i$, shown in Fig. 2). In the interval $(f, g)$ oscillatory perturbations grow and at point $g$ two monotonically increasing perturbations appear, i.e. as the Rayleigh number increases, there are at first increasing oscillatory perturbations (oscillatory instability).

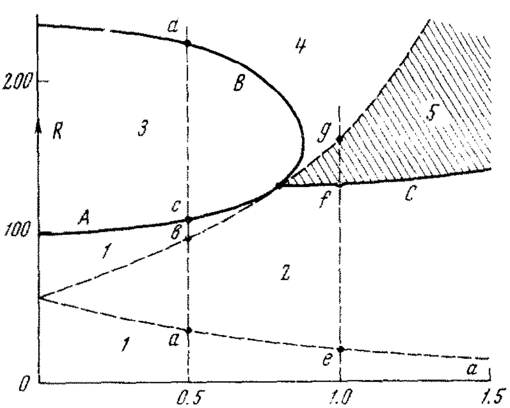


Fig. 3

When decrement spectra, similar to those shown in Figs. 1 and 2, are suitably processed, we can construct a stability chart in the plane $(R, a)$. This chart is illustrated in Fig. 3.

Vertical dashed lines correspond to the cuts ($a$ - 0.5 and 1) in which the spectra in Figs. 1 and 2 are presented. The characteristic lines are marked as follows: $A$ and $B$ are the neutral lines of monotonic perturbations, $C$ is the neutral line of oscillatory perturbations. The dashed curve bounds the region of existence of oscillatory perturbations.

The regions in the chart are as follows: 1 and 2 are the stability regions (in 1 both perturbations damp monotonically, in 2 both perturbations damp in an oscillatory manner); 3 and 4 are the monotonic instability regions (in 3 one of perturbations grows monotonically, in 4 both perturbations grow monotonically); 5 is the oscillatory instability region (both perturbations build up in an oscillatory manner).

Thus, our investigation of the nonstationary perturbation spectrum in a vertical channel with permeable boundaries leads to the conclusion that an oscillatory convective instability can exist. For small values of the Péclet number ($a < 0.8$, cf. Fig. 3), as the Rayleigh number increases, transverse motion becomes unstable with respect to monotonic perturbations, i.e. at the critical value of Rayleigh number (curve $A$) stationary convection begins to take place. For $a > 0.8$ the instability expresses itself in oscillatory perturbations; after crossing the neutral line $C$ (as $R$ increases) an oscillatory convection occurs.

Let us note in conclusion that closure of stationary levels has been detected earlier [3] in a study of convective motion stability in an inclined layer. In the present problem, the closure is accompanied not by stabilization as in [3], but by change in the mode of instability, namely by transition to oscillatory convection.

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# APPLICATION OF THE STATIONARY PHASE METHOD IN SOME PROBLEMS OF THE THEORY OF WAVES ON THE SURFACE OF A VISCOUS LIQUID

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The purpose of this paper is to develop the asymptotic representation of certain integrals encountered in the analysis of the problem of wave motion in an unbounded viscous liquid. Attention is also drawn to incorrect application of the stationary phase method widely used in a number of recent publications [2–21] dealing with the Cauchy-Poisson problem of waves on the surface of half-space or layer.

**1.** Sretenskii [1] published in 1941 a fundamental work on the subject considered in the present paper. The second Chapter of [1] deals with the two-dimensional Cauchy-Poisson problem of waves on the surface of a viscous liquid of infinite depth. By successive integral transformations of Fourier and Laplace he obtained for the first time an exact integral representation for the free surface shape. For the asymptotic calculations of the integrals obtained, the method of stationary phase was suggested. This method was developed by Kelvin and is well established in the problems of wave motions in ideal liquids.